2026 TUESDAY PROBLEMS #4

2 november

1. For a positive integer n, define

$$f(n) = \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}}$$

Show that for every positive integer k, there is some n with $f(n) = k$.

2. Let N and k be fixed positive integers.

Let g be the maximum possible gcd of a multiset of k positive integers that sum to N:

$$g = g(N,k) = \max\left\{ \gcd(x_1, x_2, \cdots, x_k) : \sum_{j=1}^{k} x_j = N, x_j \in \mathbb{N} \setminus \{0\} \right\}$$

Find g, as a function of N and k.

Is g (weakly) monotonic in N? Is g (weakly) monotonic in k?

3. Suppose that for some positive integer n, 2^n and 5^n begin with the same digit (same leftmost digit). What are the possible values of that digit?

4. Recall that the Fibonacci sequence is given by $f_0 = 1$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Show that every positive integer N divides f_n for infinitely many n.

5. Given a standard deck of 52 playing cards, assume that it has been shuffled in such a way that there are more red cards in the top 26 cards than there are black cards in the bottom 26. Prove that there are, somewhere in the deck, 3 consecutive cards of the same colour.