

2026 TUESDAY PROBLEMS #2

5 october

1. Find all integers x, y such that $x^2 - y^2 = 16$ and $x^3 - y^3 = 98$.
2. Let $N = \{1, 2, 3, \dots, n\}$.
 - a) Show that the number of subsets of N of even cardinality is equal to the number of subsets of N of odd cardinality.
 - b) Show that the number of subsets of N of even sum is equal to the number of subsets of N of odd sum.

3. Consider the group generated by the following matrices under ordinary matrix multiplication.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

In other words, this group is the set of all matrices that can be expressed as finite products of the above four matrices and their inverses, in any order, with repetitions allowed.

Does this group contain the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$?

4. Evaluate the determinants of the matrices $\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix}$.

5. Show that $\gcd(2^n - 1, 2^m - 1) = 1$ if and only if $\gcd(m, n) = 1$.

In fact more can be said: the gcd's can be related explicitly.

bonus problem:

6. Suppose that $x + 1/x$ is an integer.
 - a) Show that $x^n + 1/x^n$ is an integer for every non-negative integer n
 - b) Furthermore, find all real values of x such that $x + 1/x$ is an integer.
 - c) Furthermore, find all complex (non-real) values of x such that $x + 1/x$ is an integer.