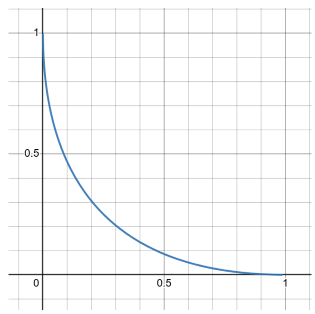
2025 TUESDAY PROBLEMS #9

19 november

1. Find all positive integers a, b, c such that $a! + b! = c!$.

For bonus marks, your proof should fit in the margin.

2. Consider the curve $\sqrt{x} + \sqrt{y} = 1$ for $x \in [0, 1]$ (and $y \in [0, 1]$ also of course). Is this an arc of a circle? If so, prove it; if not, what kind of curve is it?



3. Let A be the $n \times n$ matrix such that

$$A_{ij} = \begin{cases} 1 & j = i \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $p_n(x) = \det(xI - A)$. So $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 1$, $p_3(x) = x^3 - 2x$.

Show that $p_{n+2}(x) = xp_{n+1}(x) - p_n(x)$ for $n \geq 0$.

4. Suppose that f is a function on the interval $[1, 3]$ such that $-1 \leq f(x) \leq +1$ for all x and $\int_1^3 f(x)\,\mathrm{d}x = 0$. Determine the largest possible value of

$$\int_1^3 \frac{f(x)}{x}\,\mathrm{d}x$$

5. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a strictly decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Prove that the following integral diverges:

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,\mathrm{d}x$$