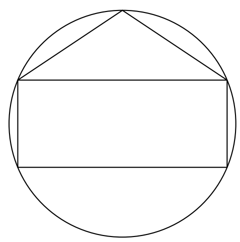
2025 TUESDAY PROBLEMS #6

8 october : Thanks, Richard!

1. Consider a brick of size $a \times b \times c$. Let B_r be the set of points $x \in \mathbb{R}^3$ such that the shortest distance between x and a point of the brick is at most r.

Find the volume of B_1.

2. Consider a rectangle of base b and height h inscribed inside a circle, and an isosceles triangle whose base is one of the sides of length b of the rectangle and whose other vertex is on the circle.



For what value of h do the rectangle and the triangle have the same area?

3. For positive integers n, let $f(n) = 1! + 2! + \cdots + n!$.

Find polynomials $p(n)$ and $q(n)$ such that $f(n+2) = p(n)f(n+1) + q(n)f(n)$ for $n \geq 1$.

4. Let $0 \leq x \leq 1$. Let the binary expansion of x be $x = x_1 2^{-1} + x_2 2^{-2} + x_3 2^{-3} \cdots$; we choose the binary expansion that does not end in infinitely many 1's (ask me if you are unsure what this is).

Define the function $f(x) = x_1 3^{-1} + x_2 3^{-2} + x_3 3^{-3} \cdots$. So, write down x in binary, but read it as if it were in ternary.

Evaluate $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$.