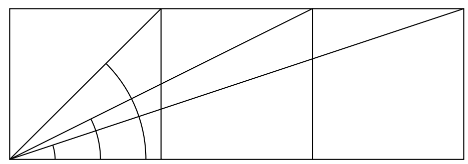
2025 TUESDAY PROBLEMS #10

26 november

1. Three squares are lined up, with diasgonals and angles as shown

Show that the angle α is the sum of angles β and γ.



2. What is the sum of all of the digits in all of the numbers from one to one billon?

3. Define $A_0 = (0)$, $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $A_{n+1} = \begin{pmatrix} A_n & I_{2^n} \\ I_{2^n} & A_n \end{pmatrix}$.

Prove that A_n has $n+1$ distinct eigenvalues with multiplicities $\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{n}$.

[Here I_m is the $m \times m$ identity matrix.]

4. Let θ be any real number with $0 < \theta < 2\pi$.

We are given an ordinary cylindrical cake with icing on top. Cut out a piece which is a wedge of angle θ (in the usual cake-cutting procedure). But then turn it upside-down and put it back in the cake. Then cut out another piece adjacent to this one, and again turn it upside-down and put it back in the cake. Keep doing this, always moving in the same orientation. After some time, each "piece" might actually be more than one piece, some of which may already be inverted, but we just continue on. Prove that after finitely many steps, all the icing will be back on top.

5. Let n be a positive integer, let $0 \le j \le n-1$, and let $f_n(j)$ be the number of subsets S of $\{0, 1, 2, \cdots, n-1\}$ such that the sum of the elements of S gives a remainder of j upon division by n. Prove or disprove: $f_n(j) \le f_n(0)$ for any $n \ge 1$ and $0 \le j \le n-1$.